PHYSICAL REVIEW E VOLUME 57, NUMBER 3 MARCH 1998

Similarities and differences between Bose and Fermi gases

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(Received 17 November 1997)

The purported ''equivalence'' between noninteracting Bose and Fermi gases in two dimensions is critically examined. Staying away from the thermodynamic limit, an exact calculation reveals some subtle, but significant, differences between the properties of the two systems that arise characteristically from the statistics obeyed by them. At the same time, certain physical quantities show differences that are not so subtle, in that they remain present even when the thermodynamic limit is taken. The role played by the spatial dimensionality d is also elucidated and it turns out that there is nothing fundamentally special about dimensionality 2; if the single-particle energy spectrum operating in the systems is of the form $\varepsilon \propto p^s$, then the equivalence in question arises when and only when $d=s$. [S1063-651X(98)11903-7]

PACS number(s): 05.30.−d, 05.70.Fh, 68.35.Md, 82.65.Dp

I. INTRODUCTION

In a recent paper Lee [1] has resurrected an old result of May [2], according to which the specific heat $C_V(N,T)$ of an ideal Bose gas in *two* dimensions is identical to that of a corresponding Fermi gas for all N, V, and T; of course, V here denotes the *area* of the two-dimensional domain to which each gas is confined. This remarkable result stems from the fact that the fugacity z_B of the Bose gas and the fugacity z_F of the Fermi gas in dimensionality 2 are mutually related through the Euler transformation

$$z_B = \frac{z_F}{1+z_F}, \tag{1a}$$

$$z_F = \frac{z_B}{1-z_B}, \tag{1b}$$

which, through a simple (yet tricky) calculation [3], shows that the internal energies U_B and U_F of the two gases differ by a mere constant, i.e.,

$$U_B(N,V,T) = U_F(N,V,T) - U_F(N,V,0). \tag{2}$$

Differentiating Eq. (2) with respect to T, at constant N and V, we readily obtain the result in question, viz.,

$$[C_V(N,T)]_B = [C_V(N,T)]_F. \tag{3}$$

Struck by the equality of the two specific heats, one is tempted to conclude that the two gases, which are normally so different, become thermodynamically equivalent when confined to two dimensions. This raises the following questions.

(i) What is so special about dimensionality 2?

(ii) As T approaches absolute zero, the Bose gas will ultimately undergo a phase transition, transferring all its particles from the excited states ($\varepsilon > 0$) to a *single* state ($\varepsilon = 0$). The Fermi gas, on the other hand, will settle into a sea of states, capped by the Fermi energy $\varepsilon_F(N/V)$. One wonders if this vital difference between the two systems would have any repercussions at finite temperatures.

(iii) How far does the ''equivalence'' of the two systems go? In other words, do statistics not matter at all, insofar as the observed properties of the two systems are concerned?

To answer these and other allied questions is the main purpose of the present paper.

In Sec. II we consider an ideal gas of bosons and side by side an ideal gas of fermions, with energy spectrum $\epsilon \propto p^s$, confined to a d-dimensional ''box'' of volume V. If d is regarded as a continuous variable (as s is), then we can readily see that the aforementioned equivalence between the two gases arises when and only when $d=s$. Thus, if the single-particle energy ε is proportional to the *square* of the momentum p (as is often the case), then the equivalence arises at $d=2$; if, on the other hand, the energy-momentum relationship is *linear* (as, for instance, happens in the case of an extreme relativistic gas), then the equivalence arises at $d=1$. Clearly, there is nothing fundamentally special about dimensionality 2.

To answer question (ii) we must consider systems of a *finite* size, for if the thermodynamic limit is invoked right from the beginning, then the phase transition in the Bose gas will set in strictly, and critically, at $T=0$ K. However, in a finite-sized system (with $d=s=2$), the same transition will be *nonsingular* and will be spread over a *range of temperatures*, whose width is determined by the precise value of N. Only then would one expect to see the real distinction between the Bose gas and the Fermi gas. The desired calculation, under periodic boundary conditions, is carried out in the Appendix and the main features of the results so obtained are discussed in Sec. III. In Sec. IV we examine question (iii) and show that there are several properties of the two systems that are significantly different from one another, not only when N and V are finite but also when N and V approach infinity.

II. SYSTEMS WITH ARBITRARY *d* AND *s*

Following the customary procedure, in which the summations-over-states appearing in the expressions for the various thermodynamic quantities pertaining to a given system are replaced by integrations, we obtain for a d-dimensional Bose gas with energy spectrum $\varepsilon = Ap^s$

$$N=\frac{\pi^{d/2}(d/s)!}{(d/2)!}\frac{V}{h^d A^{d/s}}(kT)^{d/s}g_{d/s}(z_B) \tag{4}$$

and

$$U=\frac{\pi^{d/2}(d/s)!}{(d/2)!}\frac{d}{s}\frac{V}{h^d A^{d/s}}(kT)^{(d/s)+1}g_{(d/s)+1}(z_B), \tag{5}$$

where V is the ‘‘volume’’ of the region to which the gas is confined, $g_\nu(z_B)$ are the Bose-Einstein functions, and the other symbols have their usual meanings. The corresponding expressions for the Fermi gas are

$$N=\frac{\pi^{d/2}(d/s)!}{(d/2)!}\frac{V}{h^d A^{d/s}}(kT)^{d/s}f_{d/s}(z_F) \tag{6}$$

and

$$U=\frac{\pi^{d/2}(d/s)!}{(d/2)!}\frac{d}{s}\frac{V}{h^d A^{d/s}}(kT)^{(d/s)+1}f_{(d/s)+1}(z_F), \tag{7}$$

where $f_\nu(z_F)$ are the (rival) Fermi-Dirac functions. Equating expressions (4) and (6), we obtain an implicit relationship between the fugacities z_B and z_F of the two gases, viz.,

$$g_{d/s}(z_B)=f_{d/s}(z_F). \tag{8}$$

For the special case $d=s$, this relationship assumes the simple form

$$-\ln(1-z_B)=\ln(1+z_F), \tag{9}$$

which leads to the Euler transformation (1) and in turn to Eqs. (2) and (3).

Now, keeping in mind the basic mathematical properties of the functions $g_\nu(z)$ and $f_\nu(z)$, we readily see that the aforementioned simplification and the resulting ‘‘equivalence’’ between the two systems arise only when $d=s$ and not at any other value of d. If $s=2$, this will happen in dimensionality 2; if $s=1$, it will happen in dimensionality 1. Thus, while there is nothing fundamentally special about dimensionality 2, the case $d=s$ in the present context is indeed exceptional.

We will now address question (ii) and, for this purpose, will examine critically the low-temperature behavior of the two systems. For a concrete study, we will take $d=s=2$ and, for obvious reasons, will set $V=L^2$ and $A=1/2m$.

III. LOW-TEMPERATURE BEHAVIOR OF THE GIVEN SYSTEMS WITH $d=s=2$

We start with the more intriguing case of the Bose gas for which Eqs. (4) and (5) now become

$$N=\frac{L^2}{\lambda^2}g_1(z_B) \tag{10}$$

and

$$U=\frac{L^2}{\lambda^2}kTg_2(z_B), \tag{11}$$

where λ $(=h/\sqrt{2\pi mkT})$ is the mean thermal wavelength of the particles and L the side of the box, assumed square, to which the gas is confined. At low temperatures, the fugacity z_B is very close to unity, so Eqs. (10) and (11) assume the asymptotic forms [4]

$$N\approx\frac{L^2}{\lambda^2}\ln\left(\frac{1}{\alpha}\right) \tag{12}$$

and

$$U\approx\frac{L^2}{\lambda^2}kT\left[\frac{\pi^2}{6}+\alpha\ln\alpha-\alpha\right], \tag{13}$$

where $\alpha=-\ln z_B$. Clearly, α at low temperatures is much less than unity and its temperature dependence is given by [see Eq. (12)]

$$\alpha\approx\exp(-\lambda^2/l^2), \tag{14}$$

where l $[=L/\sqrt{N}]$ is the mean interparticle distance in the system. Now the number of particles in the ground state $(\varepsilon=0)$ is given by

$$N_0=\frac{1}{z_B^{-1}-1}=\frac{1}{e^\alpha-1}\approx\frac{1}{\alpha}. \tag{15}$$

Substituting Eq. (14) into Eq. (15), we get

$$N_0\approx\exp(\lambda^2/l^2). \tag{16}$$

Unfortunately, expression (16) becomes *prohibitively large* as $T\rightarrow0$, in the sense that, while the total number of particles N, on an average, is held fixed, the number N_0, instead of approaching N, continues to rise unchecked and ultimately approaches infinity. The reason for this fallacious behavior lies in the fact that, because of the initial replacement of the summation-over-states by an integral, Eq. (10) and, along with it, Eq. (12) *do not* include particles in the condensate and account only for particles in the normal component of the system. Accordingly, the expression (14) for α and the resulting expression (16) for N_0 obtained with the help of these equations cannot be relied upon.

In fact, at low temperatures, where $\alpha\ll1$, even the normal component is not given correctly by expression (12) because, as it stands, this expression is singular in α, whereas the original sum that determines the population of this component, viz.,

$$N_e=\sum_{\varepsilon>0}(e^{\alpha+\beta\varepsilon}-1)^{-1}, \tag{17}$$

is strictly analytic in α; the true singularity of the problem rests only in the $\varepsilon=0$ term, of the full sum for N, which is absent from the partial sum (17) and stands separately in expression (15) for N_0. For the same reason, we must not succumb to the temptation of simply adding expression (15) to Eq. (12) and hoping that by doing so all will be well [5].

According to our experience, the proper way to handle this sort of problem is to carry out an exact evaluation of the full sum

$$N=\sum_{\varepsilon}\,(e^{\alpha+\beta\varepsilon}-1)^{-1} \tag{18}$$

in a manner that accounts accurately for *both* N_e and N_0. Such an evaluation is carried out in the Appendix and the desired result, for $\alpha \ll 1$, is given by Eq. (A14), which now determines α as a function of N, L, and T; note that, according to this expression, the parameter α indeed tends to the correct limit $1/N$ as $T \to 0$. Once α is known, Eq. (15) determines N_0, the number of particles in the condensate, while N_e, the number of particles in the normal component, is given by

$$N_e=\frac{L^2}{\lambda^2}\left[\ln\left(\frac{L^2}{\lambda^2}\right)+2C-\frac{y^2}{\pi}\sum_{l_{1,2}=-\infty}^{\infty}{}'\,\frac{1}{l^2(y^2+\pi^2l^2)}\right]$$

$$\left(y^2=\frac{\pi\alpha L^2}{\lambda^2},l=\sqrt{l_1^2+{}_2^2}>0\right), \tag{19}$$

C being a constant approximately equal to $-0.160\,95$; we observe that expression (19), unlike Eq. (12), is strictly analytic in α. We are now in a position to examine the region of phase transition in the system that, for a finite N, is *noncritical*.

Now the onset of phase transition in a Bose system is characterized by the fact that both N_0 and N_e are of order N. In the present case, this requires that

$$\alpha=O\left(\frac{1}{N}\right), \tag{20a}$$

and

$$\frac{\lambda^2}{l^2}=O(\ln N); \tag{20b}$$

accordingly, the parameter y^2 in the transition region would be $O(1/\ln N)$. In view of Eq. (20b), we introduce a new temperature parameter t, defined by

$$t\equiv\frac{l^2}{\lambda^2}\ln N=\left[\frac{2\pi mkl^2}{h^2}\ln N\right]T=O(1). \tag{21}$$

Equation (19) then gives for the normal fraction f_e

$$f_e\equiv\frac{N_e}{N}=t\left[1-\frac{\ln(t^{-1}\ln N)+2|C|}{\ln N}-O\left(\frac{1}{\ln N}\right)^2\right] \tag{22}$$

and hence for the condensate fraction f_0

$$f_0\equiv\frac{N_0}{N}=(1-t)+t\left[\frac{\ln(t^{-1}\ln N)+2|C|}{\ln N}+O\left(\frac{1}{\ln N}\right)^2\right], \tag{23}$$

valid for $0<t<1$, provided that both $tN/\ln N$ and $(1-t)\ln N$ are much greater than unity [6].

We thus see that in the temperature regime defined by Eq. (21), where $T \sim 1/\ln N$, the Bose gas experiences a macroscopic accumulation of particles in the ground state $\varepsilon=0$. No such accumulation is possible in the Fermi gas. Clearly, the two systems differ radically in this temperature regime. Unfortunately, the limit $N\to\infty$ makes this regime collapse onto a *single* temperature $(T=0)$ and with it the phenomenon of Bose-Einstein condensation in two dimensions becomes imperceptible.

The specific heat $C_V(N,T)$ of the Bose gas, which happened to be the main focus of both May and Lee, can be obtained from Eq. (A18) of the Appendix. We get

$$\frac{1}{k}\,C_V(N,T)=\frac{\pi^2}{3}\frac{L^2}{\lambda^2}-N\left[\alpha+T\left(\frac{\partial\alpha}{\partial T}\right)_{N,L}\right]$$

$$-\frac{L^2}{\lambda^2}\left[2\alpha+T\left(\frac{\partial\alpha}{\partial T}\right)_{N,L}\right]. \tag{24}$$

Now, by Eqs. (15) and (23),

$$\alpha\approx N_0^{-1}\approx[N(1-t)]^{-1}. \tag{25}$$

Equation (24) then takes the form

$$\frac{1}{k}\,C_V(N,T)\approx\frac{\pi^2}{3}\frac{L^2}{\lambda^2}-\frac{1}{(1-t)^2}-\frac{2-t}{(1-t)^2}\frac{l^2}{\lambda^2} \tag{26a}$$

$$=\frac{\pi^2}{3}\frac{N}{\ln N}\left[t-\frac{3}{\pi^2(1-t)^2}\frac{\ln N}{N}\right.$$

$$\left.-\frac{3t(2-t)}{\pi^2(1-t)^2}\frac{1}{N}\right]. \tag{26b}$$

By contrast, no such corrections arise in the corresponding case of the Fermi gas and we have simply the main term $\pi^2L^2/3\lambda^2$, which is directly proportional to T, as it should be for a Fermi gas. So, once again, we notice a subtle difference between the two systems in the temperature regime (21).

IV. DENSITY FLUCTUATIONS, ISOTHERMAL COMPRESSIBILITY, ETC.

In this section we consider those quantities that are significantly different for the two systems, not only when N is finite but also in the limit $N\to\infty$. We start with the isothermal compressibility κ_T, which is intimately related with the density fluctuations in the system [7]

$$\kappa_T=-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T=\frac{V}{kT}\frac{\overline{(\Delta n)^2}}{(\overline{n})^2}=-\frac{V}{(\overline{N})^2kT}\left(\frac{\partial\overline{N}}{\partial\alpha}\right)_{T,V}; \tag{27}$$

note that the quantity $\overline{N}$ here is the same as has been denoted by the simpler symbol N in the previous sections, a notation we will continue to use in the following. If we substitute the bulk expression (10) into Eq. (27), we obtain for the Bose gas

$$\kappa_T=\frac{2\pi m}{h^2n^2}\,g_0(z_B)\qquad\left(n=\frac{N}{V}=\frac{N}{L^2}\right). \tag{28}$$

In the light of the discussion carried out in Sec. III, the foregoing result would be valid only when $N_0 \ll N$. In particular, when the temperature of the gas is sufficiently high, its fugacity would be much less than unity; in that limit,

$g_0(z_B) \approx z_B \approx n\lambda^2$, with the result that κ_T would approach its classical value $1/nkT$. At low temperatures, however, where $N_0 = O(N)$, we must use expression (A14) instead of Eq. (10); the quantity κ_T then turns out to be

$$\kappa_T = \frac{L^2}{N^2 kT}\left[\frac{\pi^2 L^4}{\lambda^4} \sum_{l_{1,2}=-\infty}^{\infty}{}' \frac{1}{(y^2+\pi^2 l^2)^2} + \frac{1}{\alpha^2}\right]. \tag{29}$$

Since the parameter y^2 in this region is much less than unity, the primed sum in Eq. (29) may be approximated by

$$\frac{1}{\pi^4} \sum_{l_{1,2}=-\infty}^{\infty}{}' \frac{1}{l^4} = \frac{2G}{3\pi^2}; \tag{30}$$

here G is Catalan's constant, which is approximately equal to 0.915 966. In view of Eqs. (15) and (19), expression (29) may be written as

$$\kappa_T \approx \frac{L^2}{kT}\left[\frac{2G}{3} \frac{f_e^2}{\{\ln(L^2/\lambda^2)\}^2} + f_0^2\right]. \tag{31}$$

Clearly, the most dominant contribution to the isothermal compressibility of, and to the density fluctuations in, the Bose gas comes from the condensate fraction f_0. We may as well write

$$\kappa_T \approx (L^2/kT) f_0^2, \tag{32}$$

which shows that, in the temperature regime defined by Eq. (21), the isothermal compressibility of the Bose gas is directly proportional to the size of the system and its temperature dependence is determined by the quantity f_0^2/T. We note that results similar to Eq. (32) have been encountered earlier in the study of other finite-sized, Bose-Einstein systems undergoing a phase transition [8].

The isothermal compressibility of the Fermi gas, on the other hand, is given by the bulk expression

$$\kappa_T = \frac{2\pi m}{h^2 n^2} f_0(z_F) = \frac{2\pi m}{h^2 n^2} \frac{z_F}{1+z_F}, \tag{33}$$

which holds at *all* temperatures. For comparison with the Bose gas, we observe that when T is sufficiently low (so that z_F becomes much greater than unity), the isothermal compressibility of the Fermi gas approaches the limiting value

$$(\kappa_T)_0 = \frac{2\pi m}{h^2 n^2} = \frac{1}{n\varepsilon_F}, \tag{34}$$

where ε_F $(= nh^2/2\pi m)$ is the Fermi energy of the gas in two dimensions. The contrast between expressions (32) and (34) is indeed striking.

Next we consider the adiabatic compressibility κ_S of the two systems. For this we recall that in the case of noninteracting systems the constancy of S and N implies the constancy of the fugacity z; with $d = s$, this further implies the constancy of the product VT and of the ratio P/T^2. Consequently,

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S = -\frac{1}{V}\frac{(\partial V/\partial T)_S}{(\partial P/\partial T)_S} = -\frac{1}{V}\frac{-V/T}{2P/T} = \frac{1}{2P}. \tag{35}$$

As $T \to 0$, the adiabatic compressibility of the Bose gas diverges while that of the Fermi gas approaches the limiting value

$$(\kappa_S)_0 = \frac{1}{2P_0} = \frac{L^2}{2U_0} = \frac{1}{n\varepsilon_F}, \tag{36}$$

which is identical to $(\kappa_T)_0$. Once again, the two systems differ considerably.

Finally, in view of the thermodynamic relation $C_P/C_V = \kappa_T/\kappa_S$, we infer that while the specific heats C_V of the two systems are asymptotically equal, the specific heats C_P will be quite different. We indeed find that while the ratio C_P/C_V for the Fermi gas at low temperatures tends to unity, for the Bose gas one gets

$$\frac{C_P}{C_V} = \frac{\kappa_T}{\kappa_S} \approx \frac{(L^2/kT)f_0^2}{3\lambda^2/\pi^2 kT} = \frac{\pi^2 L^2 f_0^2}{3\lambda^2} \approx \frac{\pi^2}{3}\frac{N}{\ln N} t(1-t)^2 \gg 1; \tag{37}$$

the inequality arises from the conditions on the variables t and $1-t$ stated after Eqs. (22) and (23).

ACKNOWLEDGMENT

Financial support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

APPENDIX

In this appendix we evaluate the sum (18) for a Bose gas confined to a square box of side L. Assuming *periodic* boundary conditions, we have

$$N = \sum_{\varepsilon} (e^{\alpha+\beta\varepsilon}-1)^{-1} = \sum_{\varepsilon}\sum_{j=1}^{\infty} e^{-j\alpha - j\beta\varepsilon}, \tag{A1}$$

where

$$\varepsilon = \frac{h^2}{2mL^2}(n_1^2+n_2^2) \quad (n_{1,2} = 0, \pm 1, \pm 2, \ldots). \tag{A2}$$

In view of Eq. (A2), the sum in Eq. (A1) may be written as

$$N = \sum_{j=1}^{\infty} e^{-j\alpha} \sum_{n_{1,2}=-\infty}^{\infty} e^{-jw(n_1^2+n_2^2)} \quad \left(w = \frac{\beta h^2}{2mL^2} = \frac{\pi\lambda^2}{L^2}\right), \tag{A3}$$

where λ $(= h/\sqrt{2\pi mkT})$ is the mean thermal wavelength of the particles. Making use of the identity

$$\sum_{n=-\infty}^{\infty} e^{-an^2} = \left(\frac{\pi}{a}\right)^{1/2} \sum_{q=-\infty}^{\infty} e^{-\pi^2 q^2/a} \quad (a > 0), \tag{A4}$$

which is a special case of the Poisson summation formula, we render expression (A3) into the form

$$N = \sum_{j=1}^{\infty} e^{-j\alpha}\left(\frac{\pi}{jw}\right) \sum_{q_{1,2}=-\infty}^{\infty} e^{-\pi^2(q_1^2+q_2^2)/jw}. \tag{A5}$$

For obvious reasons, we separate the $\mathbf{q}=0$ term from the rest and write

$$N=\frac{\pi}{w}\left[\sum_{j=1}^{\infty}\frac{e^{-j\alpha}}{j}+\sum_{q_{1,2}=-\infty}^{\infty}{}'\sum_{j=1}^{\infty}\frac{e^{-j\alpha}}{j}e^{-\pi^2(q_1^2+q_2^2)/jw}\right]; \tag{A6}$$

the primed summation here implies that the term with $\mathbf{q}=0$ is excluded.

Now the summation in the first part of Eq. (A6) leads to the Bose-Einstein function $g_1(\alpha)$, while the same summation in the second part can be handled with the help of (a generalization of) the Poisson summation formula, namely [9],

$$\sum_{j=a}^{b}f(j)=\sum_{r=-\infty}^{\infty}\int_a^b f(j)e^{2\pi irj}dj+\tfrac{1}{2}f(a)+\tfrac{1}{2}f(b)\quad(b>a). \tag{A7}$$

Applying formula (A7), with $a=0$ and $b=\infty$, to the second part of expression (A6) and noting that both $f(0)$ and $f(\infty)$ vanish [because on one hand the parameter $\alpha>0$, while on the other hand the variable $(q_1^2+q_2^2)$ in *this* sum is positive definite], we get

$$N=\frac{\pi}{w}\left[g_1(\alpha)+2\sum_{q_{1,2}=-\infty}^{\infty}{}'\sum_{r=-\infty}^{\infty}K_0\left(\frac{2\pi q}{\sqrt{w}}(\alpha-2\pi ir)^{1/2}\right)\right]$$

$$(q=\sqrt{q_1^2+q_2^2}>0), \tag{A8}$$

where $K_0(z)$ is a modified Bessel function. In most practical situations, the parameter w $(=\pi\lambda^2/L^2)\ll1$. The terms with $r\neq0$ in expression (A8) are then $O(e^{-cL/\lambda})$, where c is of order unity; such terms are clearly negligible. We may therefore retain the term with $r=0$ only and write

$$N=\frac{L^2}{\lambda^2}\left[g_1(\alpha)+2\sum_{q_{1,2}=-\infty}^{\infty}{}'K_0(2yq)\right]\quad\left(y=\frac{\pi^{1/2}\alpha^{1/2}L}{\lambda}\right). \tag{A9}$$

Note that, in arriving at the foregoing result, no errors of order $(\lambda/L)^n$ are committed; in that sense, the above evaluation may be regarded as ‘‘exact.’’

It has been shown previously [10] that

$$\sum_{q_{1,2}=-\infty}^{\infty}{}'K_0(2yq)=\frac{\pi}{2y^2}+\frac{1}{2}\ln\left(\frac{y^2}{\pi}\right)+C-\frac{y^2}{2\pi}\sum_{l_{1,2}=-\infty}^{\infty}{}'\frac{1}{l^2(y^2+\pi^2l^2)}$$

$$(l=\sqrt{l_1^2+l_2^2}>0), \tag{A10}$$

where

$$C=\gamma-\ln\frac{\{\Gamma(\frac{1}{4})\}^2}{2\pi}\simeq-0.160\,95, \tag{A11}$$

γ being the well-known Euler constant. It seems important to point out here that formula (A10) is valid for *all* $y>0$, though it is particularly useful for $y\ll1$. Substituting (A10) into Eq. (A9), we obtain the desired result

$$N=\frac{L^2}{\lambda^2}\left[g_1(\alpha)+\frac{\pi}{y^2}+\ln\left(\frac{y^2}{\pi}\right)+2C-\frac{y^2}{\pi}\sum_{l_{1,2}=-\infty}^{\infty}{}'\frac{1}{l^2(y^2+\pi^2l^2)}\right]. \tag{A12}$$

For $\alpha\ll1$, which marks the onset of Bose-Einstein condensation in the system, we may use the approximation

$$g_1(\alpha)=-\ln(1-e^{-\alpha})\approx-\ln\,\alpha \tag{A13}$$

and combine it with the term $\ln(y^2/\pi)$, thus canceling the nonanalytic terms containing $\ln\alpha$. Equation (A12) then takes the form

$$N=\frac{L^2}{\lambda^2}\left[\ln\left(\frac{L^2}{\lambda^2}\right)+2C-\frac{y^2}{\pi}\sum_{l_{1,2}=-\infty}^{\infty}{}'\frac{1}{l^2(y^2+\pi^2l^2)}\right]+\frac{1}{\alpha}. \tag{A14}$$

It will be noted that the condensate N_0, which is essentially equal to $1/\alpha$, emerges naturally from the term π/y^2 in Eq. (A12). The normal component N_e is now clearly identifiable in Eq. (A14).

Following the same procedure, we obtain for the internal energy of the gas

$$U=\sum_{\varepsilon}(e^{\alpha+\beta\varepsilon}-1)^{-1}\varepsilon$$

$$=\frac{L^2}{\lambda^2}kT\left[g_2(\alpha)-2\alpha\sum_{q_{1,2}=-\infty}^{\infty}{}'K_0(2yq)\right]. \tag{A15}$$

Combining Eqs. (A9) and (A15), we obtain a rather simple result, viz.,

$$U=\frac{L^2}{\lambda^2}kT[g_2(\alpha)+\alpha g_1(\alpha)]-N\alpha kT. \tag{A16}$$

For $\alpha\ll1$,

$$g_2(\alpha)+\alpha g_1(\alpha)=\left(\frac{\pi^2}{6}+\alpha\ln\alpha-\alpha+\cdots\right)+\alpha(-\ln\alpha+\cdots)\approx\frac{\pi^2}{6}-\alpha. \tag{A17}$$

Equation (A16) then takes the form

$$U=\frac{\pi^2}{6}\frac{L^2}{\lambda^2}kT-N\alpha kT-\frac{L^2}{\lambda^2}\alpha kT. \tag{A18}$$

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[5] It so happens that this temptation works reasonably well in three dimensions because the finite-size effects in that case are no more than a mere *correction* to the bulk results. It does not work in two dimensions because in this case the finite-size effects at low temperatures turn out to be comparable to the bulk results; cf. Eq. (A14) with the sum of Eqs. (12) and (15).

[6] The first condition arises from the requirement that λ be much less than L [so that the passage from Eq. (A18) to (A19) is justified]; the second one arises from the requirement that the parameter y^2 be much less than unity [so that the last part of expression (19) may be regarded as relatively small].

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